

GENERALIZED ELECTRODYNAMICS AS A SPECIAL CASE OF METRIC INDEPENDENT STRESS THEORY

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ABSTRACT. We use a metric invariant stress theory of continuum mechanics to formulate a simple generalization of the the basic variables of electrodynamics and Maxwell's equations to general differentiable manifolds of any dimension, thus viewing generalized electrodynamics as a special case of continuum mechanics. The basic variable is the potential, or a variation thereof, which is represented as an r -form in a d -dimensional spacetime. The stress for the case of generalized electrodynamics, is assumed to be represented by a $(d - r - 1)$ -form, a generalization of the Maxwell 2-form.

1. INTRODUCTION

Metric independent, or pre-metric, aspects of electrodynamics have been studied since the beginning of the 20th century. Whittaker [Whi53, pp. 192–196], attributes the first work in this direction to Kottler [Kot22] while Truesdell and Toupin [TT60, Section F] attribute the main contribution to van Dantzig [vD34]. In recent decades, renewed interest in the subject led to further work in which notions of modern differential geometry have been utilized (see for example [HO03, Kai04, HIO06]).

In [Seg86, Seg02] we proposed a metric invariant formulation of continuum mechanics where the major objective was to introduce a metric invariant notion of stress. In particular, using the metric invariant formulation of Maxwell's equations in spacetime, and a metric invariant version of the Lorentz force, we discussed in [Seg02, Seg13] the stress energy momentum tensor of electrodynamics.

In this note, we want to use the same setting for metric invariant stress theory of continuum mechanics in order to formulate a simple generalization of the the basic variables of electrodynamics and Maxwell's equations to general differentiable manifolds of any dimension d . Thus, we present generalized electrodynamics as a special case of continuum mechanics. Here, the basic variable is the potential, or a variation thereof, which is represented as an r -form in a d -dimensional spacetime. The stress

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for the case of generalized electrodynamics, is assumed to be represented by a differential $(d - r - 1)$ -form, a generalization of the Maxwell 2-form.

We recall that for classical continuum mechanics, one assumes that the forces on a body $\mathcal{R} \subset \mathbb{R}^3$ are given in terms of a vector field \mathbf{b} defined in the physical space and a surface force $\mathbf{t}_{\mathcal{R}}$ defined on the boundary $\partial\mathcal{R}$ of \mathcal{R} . The virtual power of the forces on \mathcal{R} for a virtual velocity field w is given by

$$P_{\mathcal{R}} = \int_{\mathcal{R}} \mathbf{b} \cdot w \, dV + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}} \cdot w \, dA. \quad (1.1)$$

It is further recalled that if the dependence of $\mathbf{t}_{\mathcal{R}}$ on the body \mathcal{R} satisfies Cauchy's postulates, then, there is a 3×3 tensor field, the Cauchy tensor field σ such that $\mathbf{t}_{\mathcal{R}}(x) = \sigma(x)(\mathbf{n}(x))$ where $\mathbf{n}(x)$ is the outwards pointing normal to the boundary of \mathcal{R} at $x \in \partial\mathcal{R}$. Thus, one has

$$P_{\mathcal{R}} = \int_{\mathcal{R}} \mathbf{b} \cdot w \, dV + \int_{\partial\mathcal{R}} \sigma^T(w) \cdot \mathbf{n} \, dA \quad (1.2)$$

where the standard definition of the transpose has been used.

It is also noted that balance of moment of momentum implies traditionally that the stress tensor σ is symmetric. However, we want to examine the case where σ is skew symmetric. In this case, using the Levi-Civita symbol, we may define a vector field \mathbf{g} whose components are given by

$$\mathbf{g}_p = \frac{1}{2} \varepsilon_{pjk} \sigma_{jk}, \quad (1.3)$$

such that $\sigma^T(w) = \mathbf{g} \times w$. Thus, assuming for simplicity that $\mathbf{b} = 0$, one has

$$P_{\mathcal{R}} = \int_{\partial\mathcal{R}} (\mathbf{g} \times w) \cdot \mathbf{n} \, dA. \quad (1.4)$$

Using Gauss's theorem and the identity $\nabla \cdot (\mathbf{g} \times w) = (\nabla \times \mathbf{g}) \cdot w - \mathbf{g} \cdot (\nabla \times w)$, we have

$$P_{\mathcal{R}} = \int_{\mathcal{R}} (\nabla \times \mathbf{g}) \cdot w \, dV - \int_{\mathcal{R}} \mathbf{g} \cdot (\nabla \times w) \, dV. \quad (1.5)$$

Setting

$$\mathfrak{J} = \nabla \times \mathbf{g}, \quad \mathfrak{f} = \nabla \times w, \quad (1.6)$$

so that

$$\nabla \cdot \mathfrak{J} = 0, \quad \nabla \cdot \mathfrak{f} = 0, \quad (1.7)$$

the power may be written in the form

$$P_{\mathcal{R}} = \int_{\mathcal{R}} \mathfrak{J} \cdot w \, dV - \int_{\mathcal{R}} \mathbf{g} \cdot \mathfrak{f} \, dV. \quad (1.8)$$

We finally observe that in case we interpret w as the vector potential of magnetostatics, interpret \mathbf{g} as the magnetic field intensity, interpret \mathfrak{f} as the magnetic field and interpret \mathfrak{J} as the current density, Equations (1.61.7) are simply the restriction of Maxwell's equations to magnetostatics.

It is the generalization of this procedure to a general d -dimensional manifold that we consider below.

2. A REVIEW OF METRIC INVARIANT STRESS THEORY

We consider the following setting for metric independent continuum mechanics using the Eulerian (spatial) point of view. A smooth d -dimensional manifold \mathcal{M} will denote either the standard space manifold of continuum mechanics or space-time. Elements of a vector bundle $\xi : W \rightarrow \mathcal{M}$ will represent values of virtual generalized velocities. Virtual generalized velocities need not be of kinematic character and may involve generalized coordinates, internal degrees of freedom, order parameters, etc.

For a generic vector bundle $\pi : V \rightarrow \mathcal{M}$, we will use the notation $C^\infty(V)$ for the vector space of smooth sections. The notation $\bigwedge^r T^* \mathcal{M}$ will be used for the bundle of alternating r -tensors over \mathcal{M} . For this case, $V = \bigwedge^r T^* \mathcal{M}$ and $\Omega^r(\mathcal{M}) = C^\infty(\bigwedge^r T^* \mathcal{M})$ is the space of differential r -forms.

2.1. Generalized Force Fields on Manifolds. As is customary in continuum mechanics, forces on regions are characterized as either body forces or surface forces. For general differentiable manifolds, devoid of any Riemannian metric structure, forces are represented by the power they produce for the various virtual velocities. Let \mathcal{R} be a compact d -dimensional admissible region, a smooth orientable submanifold or a chain, in \mathcal{M} with boundary $\partial\mathcal{R}$ so that integration theory of differential forms on manifolds applies. We will make no distinction in the notation between the vector bundle ξ and its restriction to \mathcal{R} . A virtual velocity field over \mathcal{R} will be represented by a section $w : \mathcal{R} \rightarrow W$. We let $L(W, \bigwedge^p T^* \mathcal{R})$ denote the vector bundle over \mathcal{R} whose fiber at $x \in \mathcal{R}$ is the space of linear mappings $W_x \rightarrow \bigwedge^p T_x^* \mathcal{R}$ and we use the analogous notation for \mathcal{M} .

A *body force* is a section $\mathbf{b} : \mathcal{M} \rightarrow L(W, \bigwedge^d(T^* \mathcal{M}))$. Given a section w of ξ , the d -form $\mathbf{b}(w)$, given by $\mathbf{b}(w)(x) = \mathbf{b}(x)(w(x))$, represents the power density and may be integrated over \mathcal{R} . Thus, the total power expended by the body force for a velocity field $w : \mathcal{R} \rightarrow W$ on \mathcal{R} is

$$\int_{\mathcal{R}} \mathbf{b}(w). \quad (2.1)$$

A *boundary force* field, or a *surface force* field, on $\partial\mathcal{R}$ is a section

$$\mathbf{t}_{\mathcal{R}} : \partial\mathcal{R} \rightarrow L(W, \bigwedge^{d-1}(T^* \partial\mathcal{R})), \quad (2.2)$$

where again, for the sake of simplicity, we make no distinction in the notation between W and $W|_{\partial\mathcal{R}}$. Thus, for any $(d-1)$ -dimensional submanifold D of $\partial\mathcal{R}$, the power expended by the surface force for a velocity field u defined on $\partial\mathcal{R}$ is given by

$$\int_D \mathbf{t}_{\mathcal{R}}(u). \quad (2.3)$$

The total power expended by both the body force and surface force over the region \mathcal{R} and its boundary is viewed as the action of a linear functional $F_{\mathcal{R}}$ on the virtual velocity field w and is given therefore by

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \mathbf{b}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w). \quad (2.4)$$

2.2. Smooth Stress Fields on Manifolds. A *traction stress* is a section σ of $L(W, \wedge^{d-1} T^* \mathcal{M})$. Without changing the notation, we also view σ as a linear mapping

$$\sigma : C^\infty(W) \longrightarrow \Omega^{d-1}(\mathcal{M}) \quad (2.5)$$

so that for a section w , the $(d-1)$ -form $\sigma(w)$, is given $\sigma(w)(x) = \sigma(x)(w(x))$, Physically, $\sigma(w)$ is a flux field representing a flux of power. Given an oriented $(d-1)$ -submanifold $D \subset \mathcal{M}$, the $(d-1)$ -form $\sigma(w)$ may be restricted to vectors tangent to D using the inclusion $\iota_D : D \rightarrow \mathcal{M}$. Let $T\iota_D : TD \rightarrow T\mathcal{M}$ denote the tangent to the inclusion, we have a mapping, the pullback of differential forms,

$$\iota_D^* : \Omega^{d-1}(\mathcal{M}) \longrightarrow \Omega^{d-1}(D), \quad (2.6)$$

with

$$\iota_D^* \omega(x)(v_1, \dots, v_{d-1}) = \omega(x)(T_x \iota_D(v_1), \dots, T_x \iota_D(v_{d-1})), \quad v_1, \dots, v_{d-1} \in T_x D. \quad (2.7)$$

In accordance with the notation introduced above, $C^\infty(L(W, \wedge^{d-1} T^* \mathcal{M}))$ is the space of smooth stress fields and $C^\infty(L(W, \wedge^{d-1} T^* D))$ is the space of smooth surface force fields on D . We thus have a restriction mapping

$$\iota_D^* \circ \sigma : C^\infty(L(W, \wedge^{d-1} T^* \mathcal{M})) \longrightarrow C^\infty(L(W, \wedge^{d-1} T^* D)). \quad (2.8)$$

The total power expended over D for the vector field $u : D \rightarrow W|_D$ is therefore given by

$$\int_D \iota_D^*(\sigma(u)). \quad (2.9)$$

Thus, a traction stress induces a surface on any orientable hypersurface D by the generalized Cauchy formula

$$\mathbf{t} = \iota_D^* \circ \sigma. \quad (2.10)$$

Using Stokes's theorems, the power of the induced surface force $\mathbf{t}_{\mathcal{R}}$ corresponding to a generalized velocity field u on $\partial\mathcal{R}$ may be written now as

$$\begin{aligned} \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(u) &= \int_{\partial\mathcal{R}} \iota_{\partial\mathcal{R}}^*(\sigma(u)), \\ &= \int_{\mathcal{R}} d(\sigma(u)), \end{aligned} \quad (2.11)$$

so that

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \mathbf{b}_{\mathcal{R}}(w) + \mathbf{d}(\sigma(w)). \quad (2.12)$$

It is noted that the values of the d -form in the integral above are linear in the jet extension $j^1(w)$ which as section of the jet bundle $J^1(W)$. Thus, there is a section S of $L(J^1(W), \wedge^d(T^*\mathcal{M}))$ such that $\mathbf{b}(w) + \mathbf{d}(\sigma(w)) = S(j(w))$. We refer to S as the *variational stress*. This may be summarized by

$$F_{\mathcal{R}}(w) = \int_{\mathcal{R}} \mathbf{b}(w) + \mathbf{d}(\sigma(w)) = \int_{\mathcal{R}} \mathbf{b}(w) + \int_{\partial\mathcal{R}} \mathbf{t}_{\mathcal{R}}(w) = \int_{\mathcal{R}} S(j(w)) \quad (2.13)$$

which is the metric independent version of the principle of virtual work. In contrast with classical setting for stress theory in continuum mechanics, for the metric independent analysis one has to make a distinction between the traction stress σ that induces the surface traction using the generalized Cauchy formula and the variational stress which determines the density of power via the principle of virtual power. We emphasize that in the general case, since the jet of a section cannot be decomposed invariantly into the value of a section and the value of the derivative of a section, the variational stress cannot be decomposed uniquely into a component that is dual to the values of the section and a component that is dual to the values of the derivative. For example, unlike classical continuum mechanics, one cannot require that the component that is dual to the values of the section should vanish.

3. APPLICATION OF STRESS THEORY TO ELECTRODYNAMICS

In generalized electrodynamics, the foregoing analysis specializes to the case where W is the vector bundle of r -alternating tensors, $r \leq d - 1$, i.e., $W = \wedge^r T^*\mathcal{M}$. Thus, we have the sections

$$\begin{aligned} \mathbf{t}_{\mathcal{R}} &\in C^\infty \left(L(\wedge^r T^*\mathcal{M}, \wedge^{d-1} T^* \partial\mathcal{R}) \right), \\ \mathbf{b} &\in C^\infty \left(L(\wedge^r T^*\mathcal{M}, \wedge^d T^*\mathcal{M}) \right), \\ \sigma &\in C^\infty \left(L(\wedge^r T^*\mathcal{M}, \wedge^{d-1} T^*\mathcal{M}) \right). \end{aligned} \quad (3.1)$$

The sections of $W = \wedge^r T^*\mathcal{M}$ are interpreted now as variations of the potential forms (vector potential in the traditional formulation) and a generic such variation will be denoted as α (rather than w as above). The manifold \mathcal{M} is interpreted as space-time and it is natural to assume here that $\mathbf{b} = 0$.

What characterizes generalized electrodynamics is the following assumption which is of a constitutive nature: Each traction stress σ may be represented by a $(d - r - 1)$ -form \mathbf{g} , the *generalized Maxwell form*, as

$$\sigma(\alpha) = \mathbf{g} \wedge \alpha \in C^\infty \left(\wedge^{d-1} T^*\mathcal{M} \right), \quad (3.2)$$

for any section α of $\wedge^r T^* \mathcal{M}$. This assumption implies that for each region $\mathcal{R} \subset \mathcal{M}$,

$$\begin{aligned} F_{\mathcal{R}}(\alpha) &= \int_{\partial \mathcal{R}} \sigma(\alpha), \\ &= \int_{\partial \mathcal{R}} \mathbf{g} \wedge \alpha, \\ &= \int_{\mathcal{R}} d(\mathbf{g} \wedge \alpha), \\ &= \int_{\mathcal{R}} d\mathbf{g} \wedge \alpha + (-1)^{d-r-1} \int_{\mathcal{R}} \mathbf{g} \wedge d\alpha. \end{aligned} \tag{3.3}$$

Using the terminology of de Rham currents, we denote the the d -current

$$\omega \mapsto \int_{\mathcal{R}} \omega \tag{3.4}$$

simply by \mathcal{R} . The boundary of this current is the $(d-1)$ -current $\partial \mathcal{R}$ which satisfies

$$\partial \mathcal{R}(\psi) = \mathcal{R}(d\psi) \tag{3.5}$$

for any $(d-1)$ -form ψ . In addition, for a de Rham r -current T and a smooth p -form φ , with $p \leq r$, the $(r-p)$ -current $T \lrcorner \varphi$ is defined by

$$T \lrcorner \varphi(\omega) = T(\varphi \wedge \omega). \tag{3.6}$$

Thus, Equation (3.3) may be rewritten as

$$F_{\mathcal{R}}(\alpha) = \partial \mathcal{R}(\mathbf{g} \wedge \alpha) \tag{3.7}$$

so that

$$F_{\mathcal{R}} = \partial \mathcal{R} \lrcorner \mathbf{g}. \tag{3.8}$$

One may set

$$\mathbf{f} = d\alpha, \quad \mathfrak{F} = d\mathbf{g}, \tag{3.9}$$

$\mathbf{f} \in C^\infty(\wedge^{r+1} T^* \mathcal{M})$, $\mathfrak{F} \in C^\infty(\wedge^{d-r})$, so that

$$d\mathbf{f} = 0, \quad \text{and} \quad d\mathfrak{F} = 0. \tag{3.10}$$

If one views \mathbf{f} as a generalization of the Faraday form and \mathfrak{F} as a generalization of the 4-charge-current density of electrodynamics, the equations above generalize Maxwell's equations.

The total virtual power is now represented by

$$F_{\mathcal{R}}(\alpha) = \int_{\mathcal{R}} \mathfrak{F} \wedge \alpha + (-1)^{d-r-1} \mathbf{g} \wedge \mathbf{f}. \tag{3.11}$$

We may also write

$$\begin{aligned} S(j^1(\alpha)) &= \mathfrak{F} \wedge \alpha + (-1)^{d-r-1} \mathbf{g} \wedge \mathbf{f}, \\ &= \mathfrak{F} \wedge \alpha + (-1)^{d-r-1} \mathbf{g} \wedge d\alpha. \end{aligned} \tag{3.12}$$

It follows that in the case of generalized electrodynamics, the variational stress may be decomposed invariantly in the form

$$S(j^1(\alpha)) = S_0(\alpha) + S_1(d\alpha), \quad (3.13)$$

where,

$$S_0(\alpha) = \mathfrak{F} \wedge \alpha, \quad S_1(d\alpha) = (-1)^{d-r-1} \mathfrak{g} \wedge d\alpha. \quad (3.14)$$

For a $(d-r)$ -form φ , consider the de Rham r -current $\mathcal{R}_\perp \varphi$ so that

$$\mathcal{R}_\perp \varphi(\alpha) = \int_{\mathcal{R}} \varphi \wedge \alpha. \quad (3.15)$$

Thus, we may write

$$\begin{aligned} F_{\mathcal{R}}(\alpha) &= \mathcal{R}_\perp \mathfrak{F}(\alpha) + (-1)^{d-r-1} \mathcal{R}_\perp \mathfrak{g}(d\alpha), \\ &= (\mathcal{R}_\perp \mathfrak{F} + (-1)^{d-r-1} \partial(\mathcal{R}_\perp \mathfrak{g}))(\alpha), \end{aligned} \quad (3.16)$$

so that

$$F_{\mathcal{R}} = \mathcal{R}_\perp \mathfrak{F} + (-1)^{d-r-1} \partial(\mathcal{R}_\perp \mathfrak{g}). \quad (3.17)$$

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